Derivation of the Item Information Function in Any Direction for the Multidimensional Three-Parameter Logistic Model

The following is a proof for the item information function (IIF) in any direction for the Multidimensional 3-PL (M3-PL) model and location of maximum item information (θ_{max}) in a direction parallel to the discrimination vector $\boldsymbol{a}_i$. Formulas for item information and θ_{max} are well known for unidimensional 1-, 2-, and 3-Parameter Logistic (PL) models. However, IIFs and θ_{max} are not well known for compensatory, multidimensional models (Reckase & McKinley, 1991; Segall, 1996). So, the purpose of this proof is to provide explicit formulas for item information in any direction and θ_{max} in a direction parallel to the discrimination vector $\boldsymbol{a}_i$ for the Multidimensional 3-PL (M3-PL) model.

The Item Response Model

The probability of a correct response for the M3-PL model (Reckase, 1997) is

$$P_i(\theta_j) = c_i + (1 - c_i)\,[1 + \mathrm{Exp}(-L)]^{-1}, \tag{A1}$$

where $L = D(\boldsymbol{a}_i{}'\boldsymbol{\theta}_j + d_i)$, D is equal to a scaling constant 1.7 or 1, $\boldsymbol{a}_i$ is a vector of k discrimination parameters for item i, $[a_{1i}, a_{2i}, \ldots, a_{ki}]'$, k is the number of dimensions, $\boldsymbol{\theta}_j$ is a vector of k ability parameters for person j, $[\theta_{1j}, \theta_{2j}, \ldots, \theta_{kj}]'$, and d_i is a scalar related to difficulty. The probability of an incorrect response is given by $Q_i(\theta_j) = 1 - P_i(\theta_j)$ or

$$Q_i(\theta_j) = (1 - c_i)\,[1 + \mathrm{Exp}(L)]^{-1}. \tag{A2}$$

The point of steepest slope in the ability space is known as multidimensional discrimination,

$$MDISC_i = ||\boldsymbol{a}_i|| = (\boldsymbol{a}_i{}'\,\boldsymbol{a}_i)^{1/2}, \tag{A3}$$

where $\| \boldsymbol{a}_i \|$ represents the length of vector $\boldsymbol{a}_i$ that is computed as the square root of the sum of squared elements of vector $\boldsymbol{a}_i$. $MDISC_i$ is interpreted in the same manner as the discrimination parameter (a_i) in unidimensional IRT. The difficulty of the item is the signed distance from the origin of the multidimensional space to the point of steepest slope. The formula for multidimensional difficulty is given by

$$MDIFF_i = -\, d_i(\|\boldsymbol{a}_i\|)^{-1} = -\, d_i / MDISC_i, \tag{A4}$$

and it is interpreted in the same way as the difficulty parameter (b_i) in unidimensional IRT. Reckase (1985) has shown $MDIFF_i$ to be equal to the unidimensional measure of difficulty (b_i) when there is only one dimension.

Item Information in Any Direction for the M3-PL Model

The IIF in any direction for the M3-PL model (Reckase, 1997; Reckase & McKinley, 1991) is

$$I_{i\boldsymbol{u}}(\theta) = [\nabla P_i(\theta_j) \cdot \boldsymbol{u}_i\,]^2 [P_i(\theta_j) Q_i(\theta_j)]^{-1}. \tag{A5}$$

$\nabla P_i(\theta_j) \cdot \boldsymbol{u}_i$ is the directional derivative, $\nabla P_i(\theta_j)$ is the gradient. The vector of directional cosines, $\boldsymbol{u}_i$, is $[a_{1i}/\|\boldsymbol{a}_i\|, a_{2i}/\|\boldsymbol{a}_i\|, \ldots, a_{ki}/\|\boldsymbol{a}_i\|]'$ or $[\cos \alpha_{1i}, \cos \alpha_{2i}, \ldots, \cos \alpha_{ki}]'$, where $\cos \alpha_{ki}$ is the cosine of the angle (α_{ki}) from the axis orthogonal to dimension k. It should be noted that $\|\boldsymbol{u}_i\| = 1$. Next is the derivation of the information function in any direction.

The first term in brackets within (A5) is the gradient of the function $P_i(\theta_j)$, which can be written as

$$\nabla P_i(\theta_j) = [\partial P_i(\theta_j)/\partial\theta_{1j}, \partial P_i(\theta_j)/\partial\theta_{2j}, \ldots, \partial P_i(\theta_j)/\partial\theta_{kj}]', \tag{A6}$$

where $\partial P_i(\theta_j)/\partial\theta_{kj}$ is the first partial derivative of $P_i(\theta_j)$ with respect to θ_{kj}. The general expression for the derivative of the k^{th} term in the gradient is as follows:

$$\partial P_i(\theta_j)/\partial\theta_{kj} = \partial\{c_i + (1 - c_i)[1 + \mathrm{Exp}(-L)]^{-1}\}/\partial\theta_{kj} = -1(1 - c_i)[1 + \mathrm{Exp}(-L)]^{-2}\mathrm{Exp}(-L)\text{–}Da_{ki}$$

$$= Da_{ki}(1- c_i)[1 + \mathrm{Exp}(L)]^{-1}[1 + \mathrm{Exp}(-L)]^{-1}, \qquad \text{(A7)}$$

or substituting expression (2) into (7)

$$\partial P_i(\theta_j)/\partial\theta_{kj} = Da_{ki}Q_i(\theta_j)[1 + \mathrm{Exp}(-L)]^{-1}. \qquad \text{(A8)}$$

With the k elements of the gradient having the general form of (8) and the elements of $\boldsymbol{u}_i$ having the general form of $a_{ki}/\|\mathbf{a}_i\|$, the directional derivative of the function $P_i(\theta_j)$ in the direction $\boldsymbol{u}_i$ can be expressed as

$$\nabla P_i(\theta_j)\cdot\boldsymbol{u}_i = D(\boldsymbol{a}_i'\boldsymbol{u}_i)Q_i(\theta_j)[1 + \mathrm{Exp}(-L)]^{-1}. \qquad \text{(A9)}$$

With the right-hand side of (A9) substituted in (A5), the IIF for the M3-PL model in any direction is

$$I_{i\boldsymbol{u}}(\theta) = \{D(\boldsymbol{a}_i'\boldsymbol{u}_i)Q_i(\theta_j)[1 + \mathrm{Exp}(-L)]^{-1}\}^2[P_i(\theta_j)Q_i(\theta_j)]^{-1}, \qquad \text{(A10)}$$

or with some algebra, the item information in a direction $\boldsymbol{u}_i$ becomes

$$I_{i\boldsymbol{u}}(\theta) = D^2(\boldsymbol{a}_i'\boldsymbol{u}_i)^2 Q_i(\theta_j)\{P_i(\theta_j)[1 + \mathrm{Exp}(-L)]^2\}^{-1}. \qquad \text{(A11)}$$

Corollary 1a. If it is assumed that there is no guessing (i.e., $c_i = 0$), the information function in (A11) becomes the IIF in a direction for the M2-PL model,

$$I_{i\boldsymbol{u}}(\theta) = D^2(\boldsymbol{a}_i'\boldsymbol{u}_i)^2 P_i(\theta_j)\, Q_i(\theta_j), \qquad \text{(A12)}$$

which is similar to the formula derived by Reckase and McKinley (1991).

Corollary 1b. If it is assumed that discrimination parameters on all of k dimensions are fixed at 1 and there is no guessing (i.e., $\boldsymbol{a}_i = [1, 1, ..., 1]'$ and $c_i = 0$), then (A11) reduces to the M1-PL,

$$I_{i\boldsymbol{u}}(\theta) = D^2 k\, P_i(\theta_j)\, Q_i(\theta_j), \qquad \text{(A13)}$$

where k is equal to the number of dimensions.

Corollary 1c. If there is only one dimension, then (A13), (A12), and (A11) become the item information functions for the unidimensional 1-, 2-, and 3-parameter logistic models, respectively.

θ_{max} in a Direction Parallel to the Discrimination Vector $\mathbf{a}_i$ for the M3-PLM

The formula for the location of maximum item information or theta maximum in a direction parallel to the discrimination vector is derived by setting the directional derivative of the IIF for the M3-PL model equal to zero,

$$\nabla I_{i\boldsymbol{u}}(\theta) \cdot \boldsymbol{u}_i = 0, \qquad \text{(A14)}$$

where $\nabla I_{i\boldsymbol{u}}(\theta)$ is defined as $[\partial I_{i\boldsymbol{u}}(\theta)/\partial\theta_{1j}, \partial I_{i\boldsymbol{u}}(\theta)/\partial\theta_{2j}, \ldots, \partial I_{i\boldsymbol{u}}(\theta)/\partial\theta_{kj}]'$ and $\boldsymbol{u}_i$ is defined as before. The information function in (A11) is expressed in a different form as

$$I_{i\boldsymbol{u}}(\theta) = D^2(\boldsymbol{a}_i{}'\boldsymbol{u}_i)^2(1 - c_i)\text{Exp}(-L)\{[1 + c_i\text{Exp}(-L)][1 + \text{Exp}(-L)]^2\}^{-1}, \qquad \text{(A15)}$$

where L is the logit, which is equal to $D(\boldsymbol{a}_i{}'\boldsymbol{\theta_j} + d_i)$. The general form of the k^{th} element of the gradient, $\nabla I_{i\boldsymbol{u}}(\theta)$, is derived using the quotient, product, and chain rules of calculus.

$$\partial I_{i\boldsymbol{u}}(\theta) / \partial\theta_{kj} = \partial\, \mathrm{D}^2(\boldsymbol{a}_i{}'\boldsymbol{u}_i)^2(1 - c_i)\text{Exp}(-L)\{[1 + c_i\text{Exp}(-L)][1 + \text{Exp}(-L)]^2\}^{-1} / \partial\theta_{kj}$$

$$= \quad [-Da_{ki}\, D^2(\boldsymbol{a}_i{}'\boldsymbol{u}_i)^2(1 - c_i)\text{Exp}(-L)\,[1 + c_i\text{Exp}(-L)]\,[1 + \text{Exp}(-L)]^2 -$$

$$\{2[1 + \text{Exp}(-L)]\,\text{Exp}(-L) - Da_{ki}[1 + c_i\text{Exp}(-L)] + [1 + \text{Exp}(-L)]^2 c_i\text{Exp}(-L) - Da_{ki}\}\, D^2\,(\boldsymbol{a}_i{}'\boldsymbol{u}_i)^2(1 -$$

$$c_i)\text{Exp}(-L)]\{[1 + c_i\text{Exp}(-L)][1 + \text{Exp}(-L)]^2\}^{-2}$$

$$= [-a_{ki}\, D^3(\boldsymbol{a}_i{}'\boldsymbol{u}_i)^2\,[\text{Exp}(-L) - c_i\text{Exp}(-L)]\,[1 + c_i\text{Exp}(-L)]\,[1 + \text{Exp}(-L)]^2 +$$

$$a_{ki}2\mathrm{Exp}(-L)\,[1+\mathrm{Exp}(-L)]\,[1+c_i\mathrm{Exp}(-L)]\,D^3(\boldsymbol{a}_i'\boldsymbol{u}_i)^2[\mathrm{Exp}(-L)-\mathrm{c}_i\mathrm{Exp}(-L)]\;+$$

$$a_{ki}c_i\mathrm{Exp}(-L)\,[1+\mathrm{Exp}(-L)]^2\,D^3(\boldsymbol{a}_i'\boldsymbol{u}_i)^2\,[\mathrm{Exp}(-L)-c_i\mathrm{Exp}(-L)]]\{[1+c_i\mathrm{Exp}(-L)][1+\mathrm{Exp}(-L)]^2\}^{-2}$$

$$=[a_{ki}\,D^3(\boldsymbol{a}_i'\boldsymbol{u}_i)^2\,[\mathrm{Exp}(-L)-c_i\mathrm{Exp}(-L)]\,[1+\mathrm{Exp}(-L)]^2\{-1-c_i\mathrm{Exp}(-L)+c_i\mathrm{Exp}(-L)\}+$$

$$a_{ki}2\mathrm{Exp}(-L)\,[1+\mathrm{Exp}(-L)]\,[1+c_i\mathrm{Exp}(-L)]\,D^3(\boldsymbol{a}_i'\boldsymbol{u}_i)^2[\mathrm{Exp}(-L)-c_i\mathrm{Exp}(-L)]]\;\{[1+c_i\mathrm{Exp}(-L)][1+$$

$$\mathrm{Exp}(-L)]^2\}^{-2}$$

$$=[a_{ki}\,D^3(\boldsymbol{a}_i'\boldsymbol{u}_i)^2\,[\mathrm{Exp}(-L)-c_i\mathrm{Exp}(-L)]\,[1+\mathrm{Exp}(-L)]^2\{-1\}+a_{ki}2\mathrm{Exp}(-L)\,[1+\mathrm{Exp}(-L)]$$

$$[1+c_i\mathrm{Exp}(-L)]\,D^3(\boldsymbol{a}_i'\boldsymbol{u}_i)^2[\mathrm{Exp}(-L)-c_i\mathrm{Exp}(-L)]]\{[1+c_i\mathrm{Exp}(-L)][1+\mathrm{Exp}(-L)]^2\}^{-2}$$

$$=D^3a_{ki}(\boldsymbol{a}_i'\boldsymbol{u}_i)^2\,[\mathrm{Exp}(-L)-c_i\mathrm{Exp}(-L)]\,[1+\mathrm{Exp}(-L)]\,[\{-1\}[1+\mathrm{Exp}(-L)]+2\mathrm{Exp}(-L)$$

$$[1+c_i\mathrm{Exp}(-L)]\,]\{[1+c_i\mathrm{Exp}(-L)][1+\mathrm{Exp}(-L)]^2\}^{-2}$$

$$=D^3a_{ki}(\boldsymbol{a}_i'\boldsymbol{u}_i)^2\,[\mathrm{Exp}(-L)-c_i\mathrm{Exp}(-L)]\,[1+\mathrm{Exp}(-L)]\,[\{-1\}[1+\mathrm{Exp}(-L)]+2\mathrm{Exp}(-L)$$

$$[1+c_i\mathrm{Exp}(-L)]\,]\{\,[1+c_i\mathrm{Exp}(-L)]^2[1+\mathrm{Exp}(-L)]^4\}^{-1}$$

$$=D^3a_{ki}(\boldsymbol{a}_i'\boldsymbol{u}_i)^2\,[\mathrm{Exp}(-L)-c_i\mathrm{Exp}(-L)]\,[\{-1\}[1+\mathrm{Exp}(-L)]+2\mathrm{Exp}(-L)$$

$$[1+c_i\mathrm{Exp}(-L)]\,]\{[1+c_i\mathrm{Exp}(-L)]^2[1+\mathrm{Exp}(-L)]^3\}^{-1}$$

$$=\;-D^3a_{ki}(\boldsymbol{a}_i'\boldsymbol{u}_i)^2[\mathrm{Exp}(-L)-c_i\mathrm{Exp}(-L)]\,[1+\mathrm{Exp}(-L)]\{[1+c_i\mathrm{Exp}(-L)]^2[1+\mathrm{Exp}(-L)]^3\}^{-1}$$

$$+\,D^3a_{ki}(\boldsymbol{a}_i'\boldsymbol{u}_i)^2\,[\mathrm{Exp}(-L)-c_i\mathrm{Exp}(-L)]\,2\mathrm{Exp}(-L)\,[1+c_i\mathrm{Exp}(-L)]\{[1+c_i\mathrm{Exp}(-L)]^2[1+\mathrm{Exp}(-L)$$

$$]^3\}^{-1}$$

$$=\;-D^3a_{ki}(\boldsymbol{a}_i'\boldsymbol{u}_i)^2\,[\mathrm{Exp}(-L)-c_i\mathrm{Exp}(-L)]\{[1+c_i\mathrm{Exp}(-L)]^2[1+\mathrm{Exp}(-L)]^2\,\}^{-1}\;+$$

$$D^3a_{ki}(\boldsymbol{a}_i'\boldsymbol{u}_i)^2\,[\mathrm{Exp}(-L)-c_i\mathrm{Exp}(-L)]\,2\mathrm{Exp}(-L)\{[1+c_i\mathrm{Exp}(-L)]\,[1+\mathrm{Exp}(-L)]^3\}^{-1}. \qquad \text{(A16)}$$

From (A16), the k^{th} element of the gradient $\nabla\, I_{i\boldsymbol{u}}(\theta)$ can be expressed as

$$\partial I_{i\boldsymbol{u}}(\theta) / \partial\theta_{kj} = [D^3 a_{ki}(\boldsymbol{a}_i{}' \boldsymbol{u}_i)^2 [\mathrm{Exp}(-L) - c_i\mathrm{Exp}(-L)]\{[1 + c_i\mathrm{Exp}(-L)] [1 + \mathrm{Exp}(-L)]^2\}^{-1}]$$

$$\{-1[1 + c_i\mathrm{Exp}(-L)]^{-1} + 2\mathrm{Exp}(-L)[1 + \mathrm{Exp}(-L)]^{-1}\}. \quad \text{(A17)}$$

With the k elements of the gradient, $\nabla I_{i\boldsymbol{u}}(\theta)$, having the general form of (A17) and the elements of $\boldsymbol{u}_i$ have the general form of $a_{ki}/\|\boldsymbol{a}_i\|$, the directional derivative of the IIF, $I_{i\boldsymbol{u}}(\theta)$, in the direction $\boldsymbol{u}_i$, which is parallel to $\boldsymbol{a}_i$, is expressed as

$$\nabla I_{i\boldsymbol{u}}(\theta) \cdot \boldsymbol{u}_i = [D^3(\boldsymbol{a}_i{}' \boldsymbol{u}_i)^3 [\mathrm{Exp}(-L) - c_i\mathrm{Exp}(-L)]\{[1 + c_i\mathrm{Exp}(-L)] [1 + \mathrm{Exp}(-L)]^2\}^{-1}]$$

$$\{-1[1 + c_i\mathrm{Exp}(-L)]^{-1} + 2\mathrm{Exp}(-L)[1 + \mathrm{Exp}(-L)]^{-1}\}= 0. \quad \text{(A18)}$$

From (A18), item information is maximized when the following condition is satisfied:

$$-1[1 + c_i\mathrm{Exp}(-L)]^{-1} + 2\mathrm{Exp}(-L)[1 + \mathrm{Exp}(-L)]^{-1}= 0, \quad \text{(A19)}$$

or

$$P_i(\theta) = .5\mathrm{Exp}(L). \quad \text{(A20)}$$

Expression (A20) is a sufficient condition for maximizing item information for the unidimensional and multidimensional 1-, 2-, and 3-PL models. In the unidimensional 1-PL model, information is maximized when $P_i(\theta) = Q_i(\theta) = .5$. When $P_i(\theta)$ is equal to .5, then the natural logarithm of the odds of getting the item correct is 0, i.e., $\ln[P_i(\theta)/Q_i(\theta)] = 0$, which implies that L in (A20) is 0 and $\mathrm{Exp}(L) = 1$. This leaves the well-known condition for maximizing information in the 1- and 2-PL cases, which is $P_i(\theta) = Q_i(\theta) = .5$. In classical test theory, this is akin to maximizing item variance at $p_i = .5$. For the 3-PL unidimensional and multidimensional models, the right-hand side of (A20) and the probability of a correct response are not equal to .5 when information is maximized, but the equality is satisfied at a different value, which is primarily a function of guessing. The expression in (A20) can also be written as

$$2\mathrm{Exp}(-L) = [P_i(\theta)]^{-1}. \qquad \text{(A21)}$$

The probability of a correct response can also be written as

$$P_i(\theta) = [\mathrm{Exp}(L) + c_i][\mathrm{Exp}(L) + 1]^{-1}. \qquad \text{(A22)}$$

With the right-hand side of (A22) substituted into (A21),

$$2\mathrm{Exp}(-L) = [\mathrm{Exp}(L) + 1][\mathrm{Exp}(L) + c_i]^{-1}. \qquad \text{(A23)}$$

To solve for L, expression (A23) is written as

$$2 + 2c_i\mathrm{Exp}(-\mathrm{L}) = \mathrm{Exp}(L) + 1. \qquad \text{(A24)}$$

After multiplying both sides by Exp(L),

$$2\mathrm{Exp}(L) + 2c_i = \mathrm{Exp}(2L) + \mathrm{Exp}(L), \qquad \text{(A25)}$$

which, after subtracting 2Exp(L) from both sides, is equivalent to

$$2c_i = \mathrm{Exp}(2L) - \mathrm{Exp}(L). \qquad \text{(A26)}$$

After multiplying both sides by 4, expression (A26) becomes

$$8c_i = 4\,\mathrm{Exp}(2L) - 4\mathrm{Exp}(L). \qquad \text{(A27)}$$

When 1 is added to both sides, then (A27) becomes

$$8c_i + 1 = 4\,\mathrm{Exp}(2L) - 4\mathrm{Exp}(L) + 1. \qquad \text{(A28)}$$

By way of the binomial theorem, the equality of (A28) can be written as

$$8\mathrm{c}_\mathrm{i} + 1 = [2\,\mathrm{Exp}(L) - 1]^2, \qquad \text{(A29)}$$

which after taking the square root of both sides becomes

$$(8c_i + 1)^{1/2} = 2\,\mathrm{Exp}(L) - 1. \qquad \text{(A30)}$$

Solving for L in (A30) gives

$$L = \ln\{.5\,[1 + (8c_i + 1)^{1/2}]\}, \qquad \text{(A31)}$$

which becomes

$$D(\boldsymbol{a}_i{}'\theta + d_i) = \ln\{.5\,[1 + (8c_i + 1)^{1/2}\,]\}. \tag{A32}$$

Now the objective is to solve for the vector θ that maximizes the information function of the M3-PL model in the direction $\boldsymbol{a}_i$. To thist end,

$$\boldsymbol{a}_i{}'\theta = \ln\{.5\,[1 + (8c_i + 1)^{1/2}\,]\}D^{-1} - d_i. \tag{A33}$$

Because $\boldsymbol{u}_i{}' = \boldsymbol{a}_i{}'/||\boldsymbol{a}_i||$, both sides are divided by $||\boldsymbol{a}_i||$ or $MDISC_i$ that results in

$$\boldsymbol{u}_i{}'\theta = \ln\{.5\,[1 + (8c_i + 1)^{1/2}\,]\}(D||\boldsymbol{a}_i||)^{-1} - d_i(||\boldsymbol{a}_i||)^{-1}, \tag{A34}$$

Now both sides are pre-multiplied by $\boldsymbol{u}_i$,

$$\boldsymbol{u}_i\,\boldsymbol{u}_i{}'\,\theta = \boldsymbol{u}_i[\ln\{.5\,[1 + (8c_i + 1)^{1/2}\,]\}(D||\boldsymbol{a}_i||)^{-1} - d_i(||\boldsymbol{a}_i||)^{-1}]. \tag{A35}$$

Because the associative law holds for multiplication of matrices, the left-hand side of (A35) can be written as $(\boldsymbol{u}_i\,\boldsymbol{u}_i{}')\theta$. The product of $(\boldsymbol{u}_i\,\boldsymbol{u}_i{}')$ is a k by k matrix, $\boldsymbol{U}$, which has a few properties that should be noted: 1. It is symmetric, thus $\boldsymbol{U} = \boldsymbol{U}'$, 2. The determinant of the matrix $\boldsymbol{U}$ is zero (0), thus it is singular, 3. $\boldsymbol{U}^2 = \boldsymbol{U}'\boldsymbol{U} = \boldsymbol{U}\boldsymbol{U} = \boldsymbol{U}$, thus $\boldsymbol{U}$ is idempotent, 4. The main diagonal elements of $\boldsymbol{U}$ are $\cos^2\alpha_{ki}$, thus the trace of $\boldsymbol{U}$ [i.e., tr($\boldsymbol{U}$)] is equal to one (1), and 5. The rank of an idempotent matrix is equal to its trace, thus the rank of $\boldsymbol{U}$ is equal to 1. With $\boldsymbol{U}$ substituted for $\boldsymbol{u}_i\,\boldsymbol{u}_i{}'$, expression (A35) is written as

$$\boldsymbol{U}\theta = \boldsymbol{u}_i[\ln\{.5\,[1 + (8c_i + 1)^{1/2}\,]\}(D||\boldsymbol{a}_i||)^{-1} - d_i(||\boldsymbol{a}_i||)^{-1}]\,. \tag{A36}$$

Pre-multiply both sides by $\boldsymbol{U}$ yields

$$\boldsymbol{U}\boldsymbol{U}\theta = \boldsymbol{U}\boldsymbol{u}_i[\ln\{.5\,[1 + (8c_i + 1)^{1/2}\,]\}(D||\boldsymbol{a}_i||)^{-1} - d_i(||\boldsymbol{a}_i||)^{-1}]. \tag{A37}$$

By the third property mentioned above for $\boldsymbol{U}$, (A37) can be written as

$$\boldsymbol{U}\theta = \boldsymbol{U}\boldsymbol{u}_i[\ln\{.5\,[1 + (8c_i + 1)^{1/2}\,]\}(D||\boldsymbol{a}_i||)^{-1} - d_i(||\boldsymbol{a}_i||)^{-1}]. \tag{A38}$$

Expression (A38) is written as

$$\boldsymbol{U}\theta \;-\; \boldsymbol{U}\boldsymbol{u}_i[\ln\{.5\,[1 + (8c_i + 1)^{1/2}\,]\}(D\|\boldsymbol{a}_i\|)^{-1} - d_i(\|\boldsymbol{a}_i\|)^{-1}] = \mathbf{0}, \tag{A39}$$

where $\mathbf{0}$ is the same size as $\boldsymbol{u}_i$ or by the distributive law for matrices,

$$\boldsymbol{U}\{\theta \;-\; \boldsymbol{u}_i[\ln\{.5\,[1 + (8c_i + 1)^{1/2}\,]\}(D\|\boldsymbol{a}_i\|)^{-1} - d_i(\|\boldsymbol{a}_i\|)^{-1}]\} = \mathbf{0}. \tag{A40}$$

Solutions that satisfy (A40) are when $\boldsymbol{U}$ is equal to a null matrix ($\boldsymbol{O}$) and when

$$\theta = \boldsymbol{u}_i[\ln\{.5\,[1 + (8c_i + 1)^{1/2}\,]\}(D\|\boldsymbol{a}_i\|)^{-1} - d_i(\|\boldsymbol{a}_i\|)^{-1}]. \tag{A41}$$

Therefore, theta maximum in a specified direction for the M3-PL model is

$$\theta_{\max} = \boldsymbol{u}_i[\ln\{.5\,[1 + (8c_i + 1)^{1/2}\,]\}(D\|\boldsymbol{a}_i\|)^{-1} - d_i(\|\boldsymbol{a}_i\|)^{-1}] \tag{A42}$$

or substituting (A3) and (A4) into (A42)

$$\theta_{\max} = \boldsymbol{u}_i[\ln\{.5\,[1 + (8c_i + 1)^{1/2}\,]\}(D \cdot MDISC_i)^{-1} + MDIFF_i]. \tag{A43}$$

Several corollaries follow from this result in (A43).

Corollary 2a. The location on the k^{th} dimension where information is maximized is given by

$$\theta_{\max\,k} = [\ln\{.5\,[1 + (8c_i + 1)^{1/2}\,]\}(D \cdot MDISC_i)^{-1} + MDIFF_i] \cos \alpha_{ki}\,. \tag{A44}$$

Corollary 2b. If it is assumed that there is no guessing, i.e., $c_i = 0$, expression (A43) reduces to

$$\theta_{\max} = MDIFF_i\, \boldsymbol{u}_i. \tag{A45}$$

Expression (A45) is the same as that implied by Reckase and McKinley (1991) for the location of maximum item information for the multidimensional 2-PL model.

Corollary 2c. If it is assumed that there is only one dimension, then the expression in (A43) reduces to the well-known formula for theta maximum derived by Birnbaum (1968),

$$\theta_{\max} = \ln\{.5\,[1 + (8c_i + 1)^{1/2}\,]\}(Da_i)^{-1} + b_i\,. \tag{A46}$$

IIFs and formulas for the location of maximum item information for the multidimensional 1-, 2-, and 3-PL models are listed in Table 2.

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Table 1.
Item Response Functions, Item Information Functions, and Theta Maximums for Dichotomous, Unidimensional 1-, 2-, and 3-Parameter Logistic Models

Model (Unidimensional)	Item Information Function
	Theta Maximum
1- Parameter Logistic $P_i(\theta_j) = \{1 + \text{Exp}[-D(\theta_j - b_i)]\}^{-1}$	$I_i(\theta) = D^2 P_i(\theta)Q_i(\theta)$
	$\theta_{max} = b_i$
2- Parameter Logistic $P_i(\theta_j) = [1 + \text{Exp}(-L)]^{-1}$	$I_i(\theta) = D^2 a_i^2 P_i(\theta)Q_i(\theta)$
	$\theta_{max} = b_i$
3- Parameter Logistic $P_i(\theta_j) = c_i + (1 - c_i)[1 + \text{Exp}(-L)]^{-1}$	$I_i(\theta) = D^2 a_i^2 Q_i(\theta)\{P_i(\theta)[1 + \text{Exp}(-L)]^2\}^{-1}$
	$\theta_{max} = \ln\{.5[1 + (1 + 8c_i)^{1/2}]\}/Da_i + b_i$

Note: D is equal to a scaling constant 1.7 or 1, a_i = discrimination of item i, b_i = difficulty of item i, c_i = guessing of item i, $\text{Exp}(-L) = 2.71828^{-L}$, $L = Da_i(\theta - b_i)$, and $Q_i(\theta) = 1 - P_i(\theta)$.

Table 2.
Item Response Functions, Directional Item Information Functions, and Theta Maximums for Dichotomous, Multidimensional 1-, 2-, and 3-Parameter Logistic Models

Model (Multidimensional)	Directional Item Information Function
	Theta Maximum
1- Parameter Logistic $P_i(\theta_j) = \{1 + \text{Exp}[-D(\mathbf{1}'\theta_j + d_i)]\}^{-1}$	$I_{i\boldsymbol{u}}(\theta) = D^2 k P_i(\theta) Q_i(\theta)$
	$\theta_{\text{max}} = [MDIFF_i\,/(k)^{1/2}, \ldots, MDIFF_i\,/(k)^{1/2}]'$
2- Parameter Logistic $P_i(\theta_j) = [1 + \text{Exp}(-L)]^{-1}$	$I_{i\boldsymbol{u}}(\theta) = D^2(\boldsymbol{a}_i'\,\boldsymbol{u}_i)^2 P_i(\theta) Q_i(\theta)$
	$\theta_{\text{max}} = [MDIFF_i \cos\alpha_{1i}, \ldots, MDIFF_i \cos\alpha_{ki}]'$
3- Parameter Logistic $P_i(\theta_j) = c_i + (1 - c_i)[1 + \text{Exp}(-L)]^{-1}$	$I_{i\boldsymbol{u}}(\theta) = D^2(\boldsymbol{a}_i'\,\boldsymbol{u}_i)^2\, Q_i(\theta)\{P_i(\theta)\,[1 + \text{Exp}(-L)]^2\}^{-1}$
	$\theta_{\text{max}} = \big[\ln\{.5\,[1 + (8c_i + 1)^{1/2}]\}(D\cdot MDISC_i)^{-1} + MDIFF_i\big]\boldsymbol{u}_i$

Note: k =number of dimensions, **1** is a k x 1 vector of ones. D is equal to a scaling constant 1.7 or 1, $\boldsymbol{a}_i$ is a k x 1 vector of discrimination parameters for item i, $[a_{1i}, a_{2i}, \ldots, a_{ki}]'$, θ is a vector of k ability parameters, $[\theta_{1j}, \theta_{2j}, \ldots, \theta_{kj}]'$, d_i is a scalar related to difficulty, $L = D(\boldsymbol{a}_i'\theta + d_i)$, $MDISC_i = \|\boldsymbol{a}_i\|$, $MDIFF_i = -d_i\,/\|\boldsymbol{a}_i\|$, and $\boldsymbol{u}$ is a vector of directional cosines, $\boldsymbol{a}_i\,/\|\boldsymbol{a}_i\|$ or $[\cos\alpha_{1i}, \cos\alpha_{2i}, \ldots, \cos\alpha_{ki}]'$.