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A NOTE ON ITEM INFORMATION IN ANY DIRECTION FOR THE MULTIDIMENSIONAL THREE-PARAMETER LOGISTIC MODEL

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The purpose of this note is twofold: (a) to present the formula for the item information function (IIF) in any direction for the Multidimensional 3-Parameter Logistic (M3-PL) model and (b) to give the equation for the location of maximum item information ($\boldsymbol{\theta}_{\max}$) in the direction of the item discrimination vector. Several corollaries are given. Implications for future research are discussed.

Key words: item information, measurement direction, multidimensional measurement, maximum information, three-parameter logistic model

Introduction

The item information function (IIF) and the location of maximum item information ($\theta_{\max}$) play important roles in modern test development (Birnbaum, 1968; Hambleton & Swaminathan, 1985; Samejima, 1977); this is especially the case when constructing criterion-referenced and computer-adaptive tests (Weiss, 1982). Formulas for IIFs and $\theta_{\max}$ are known for unidimensional models (Hambleton & Swaminathan, 1985) but are not well known or understood for compensatory, multidimensional models (Reckase, 1997; Segall, 1996).

In general, multidimensional extensions of IIFs are given as matrices (Segall, 1996). However, if the IIF is represented as a scalar-valued function instead of a matrix, the process of selecting items for criterion-referenced or computer-adaptive tests in a multidimensional context may be simplified. Notwithstanding the work by Reckase and McKinley (1991), little research has been conducted in the area of scalar IIFs for multidimensional models. Reckase and McKinley (1991) have derived the IIF in any direction for the Multidimensional 2-Parameter Logistic (M2-PL) model. So, the purpose of this research is twofold: (a) to present the IIF in any direction for the Multidimensional 3-PL (M3-PL) model and (b) to give the formula for $\boldsymbol{\theta}_{\max}$ in the direction of the item discrimination vector $\mathbf{a}_{\mathrm{i}}$. First, the M3-PL model is described, which follows the basic notation and definitions of Reckase (1997). The formula for the IIF in any direction is given, and the equation for $\boldsymbol{\theta}_{\max}$ in the direction of the item discrimination vector $\mathbf{a}_{\mathrm{i}}$ is then presented.

The probability of a correct response for the M3-PL model (Reckase, 1997) is

$$P_i(\boldsymbol{\theta}_j) = c_i + (1 - c_i)[1 + \exp(-L)]^{-1}, \tag{1}$$

where $L = D(\mathbf{a}_i'\boldsymbol{\theta}_j + d_i)$, D is equal to a scaling constant 1.7 or 1, $\mathbf{a}_i$ is a vector of k discrimination parameters for item i, $[a_{1j}, a_{2j}, \ldots, a_{ki}]$, k is the number of dimensions, $\boldsymbol{\theta}_j$ is a vector of k ability parameters for person j, $[\theta_{1j}, \theta_{2j}, \ldots, \theta_{kj}]$, d_i is a scalar related to difficulty, and c_i is the guessing parameter. The probability of an incorrect response is given as $Q_i(\boldsymbol{\theta}_j) = 1 - P_i(\boldsymbol{\theta}_j)$. Analogous to the point of steepest slope in unidimensional IRT, multidimensional discrimination is

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$$\text{MDISC}_i = \|\mathbf{a}_i\| = \left(\mathbf{a}_i' \, \mathbf{a}_i\right)^{1/2}, \tag{2}$$

where $\| \|$ represents the length of a vector, which is the square root of the sum of squared elements of a vector. MDISC is interpreted in the same manner as the discrimination parameter ($\underline{a}_i$) in unidimensional IRT. The difficulty of the item is the signed distance from the origin to the point of steepest slope. The formula for multidimensional difficulty is

$$\text{MDIFF}_i = -d_i(\|\mathbf{a}_i\|)^{-1} = \frac{-d_i}{\text{MDISC}_i}, \tag{3}$$

and it is interpreted similar to the difficulty parameter (b_i) in unidimensional IRT.

Item Information Function in Any Direction for the M3-PL Model

The IIF of item i in any direction $\mathbf{u}$ for the M3-PL model (Reckase, 1997) is

$$I_{i\mathbf{u}}(\boldsymbol{\theta}_j) = \frac{\left[\nabla P_i(\boldsymbol{\theta}_j) \cdot \mathbf{u}_i\right]^2}{\left[P_i(\boldsymbol{\theta}_j) Q_i(\boldsymbol{\theta}_j)\right]}. \tag{4}$$

$\nabla P_i(\boldsymbol{\theta}_j) \cdot \mathbf{u}_i$ is the directional derivative of $P_i(\boldsymbol{\theta}_j)$; $\nabla P_i(\boldsymbol{\theta}_j)$ is the gradient. The vector of directional cosines is given by $\mathbf{u}_i = [\cos\alpha_{1i}, \cos\alpha_{2i}, \ldots, \cos\alpha_{ki}]'$, and $\cos\alpha_{ki}$ is the cosine of the angle (α_{ki}) between the item vector and the axis orthogonal to dimension k.

Although explicit computations and proofs are not provided, the technical report with mathematical derivations can be obtained at http://pegasus.cc.ucf.edu/dbryant/Report_1.pdf. The first term in brackets within (4) is the gradient of the function $P_i(\boldsymbol{\theta}_j)$:

$$\nabla P_i(\boldsymbol{\theta}_j) = \left[\frac{\partial P_i(\boldsymbol{\theta}_j)}{\partial \theta_{1j}}, \frac{\partial P_i(\boldsymbol{\theta}_j)}{\partial \theta_{2j}}, \ldots, \frac{\partial P_i(\boldsymbol{\theta}_j)}{\partial \theta_{kj}}\right], \tag{5}$$

where $\partial P_i(\boldsymbol{\theta}_j)/\partial\theta_{kj}$ is the first partial derivative of $P_i(\boldsymbol{\theta}_j)$ with respect to θ_{kj}. The directional derivative of (1) is set equal to zero, and the result is substituted into (4).

Proposition 1. The IIF in any direction $\mathbf{u}$ for the M3-PL model is expressed as

$$I_{i\mathbf{u}}(\boldsymbol{\theta}_j) = D^2 \left(\mathbf{a}_i' \, \mathbf{u}_i\right)^2 Q_i(\boldsymbol{\theta}_j) \left\{P_i(\boldsymbol{\theta}_j)[1 + \exp(-L)]^2\right\}^{-1}. \tag{6}$$

Corollary 1a. If it is assumed that there is no guessing (i.e., $c_i = 0$), the information function in (6) becomes the IIF in any direction for the M2-PL model,

$$I_{i\mathbf{u}}(\boldsymbol{\theta}_j) = D^2 \left(\mathbf{a}_i' \, \mathbf{u}_i\right)^2 P_i(\boldsymbol{\theta}_j) Q_i(\boldsymbol{\theta}_j), \tag{7}$$

which is similar to the formula derived by Reckase and McKinley (1991).

Corollary 1b. If it is assumed that the discrimination parameters on all of k dimensions are fixed at 1 and there is no guessing (i.e., $\mathbf{a}_i = [1, 1, \ldots, 1]'$ and $c_i = 0$), then (6) reduces to the M1-PL,

$$I_{i\mathbf{u}}(\boldsymbol{\theta}_j) = D^2 k P_i(\boldsymbol{\theta}_j) Q_i(\boldsymbol{\theta}_j). \tag{8}$$

Corollary 1c. If there is only one dimension, then (8), (7), and (6) become the IIFs for the unidimensional 1-, 2-, and 3-PL models, respectively.

$\boldsymbol{\theta}_{\max}$ *in the Direction of the Item Discrimination Vector*

The location of maximum item information or $\boldsymbol{\theta}_{\max}$ in the direction of the item discrimination vector is derived by setting the directional derivative of (6) equal to zero,

$$\nabla I_{i\mathbf{u}}(\boldsymbol{\theta}_j) \cdot \mathbf{u}_i = 0, \tag{9}$$

where $\nabla I_{i\mathbf{u}}(\boldsymbol{\theta}_j)$ is defined as $[\partial I_{i\mathbf{u}}(\boldsymbol{\theta}_j)/\partial\theta_{1j}, \partial I_{i\mathbf{u}}(\boldsymbol{\theta}_j)/\partial\theta_{2j}, \ldots, \partial I_{i\mathbf{u}}(\boldsymbol{\theta}_j)/\partial\theta_{kj}]'$ and $\mathbf{u}_i$ is defined as before. The directional derivative of (6) is

$$\nabla I_{i\mathbf{u}}(\boldsymbol{\theta}_j) \cdot \mathbf{u}_i = \left[D^3(\mathbf{a}_i\mathbf{u}_i)^3[\exp(-L) - c_i\exp(-L)]\{[1 + c_i\exp(-L)][1+\exp(-L)]^2\}^{-1}\right]$$
$$\left\{-1\left[1 + c_i\exp(-L)\right]^{-1} + 2\exp(-L)[1+\exp(-L)]^{-1}\right\}. \tag{10}$$

From (10), when item information is maximized, the following condition will be satisfied:

$$P_i(\boldsymbol{\theta}_j) = 0.5\exp(L). \tag{11}$$

In order to find the point of maximum item information in the direction parallel to $\mathbf{a}_i$, one approach is to solve for the logit [L, which is $D(\mathbf{a}_i{}'\boldsymbol{\theta}_j + d_i)$] and then solve for $\boldsymbol{\theta}$. One begins the algebra by substituting (1) into (11) and solving for the logit, which yields

$$D\left(\mathbf{a}_i{}'\boldsymbol{\theta}_j + d_i\right) = \ln\left\{0.5\left[1 + (8c_i + 1)^{1/2}\right]\right\}. \tag{12}$$

The objective is to solve for the vector $\boldsymbol{\theta}$ in (12) that maximizes the information function of the M3-PL model in the direction $\mathbf{u}_i$, which is parallel to $\mathbf{a}_i$. It should be noted that in general, the solution space for (12) should be an affine $k-1$ dimensional space, because (12) is an affine linear system for the unknown k-dimensional vector $\boldsymbol{\theta}_j$ with rank 1 coefficient matrix $\mathbf{a}_i$. For the purpose of this study, interest is only in the elements of that space parallel to the discrimination vector $\mathbf{a}_i$. It can be verified by substitution into (12) that the maximizer in this direction is

$$\boldsymbol{\theta} = \mathbf{u}_i\left[\ln\left\{0.5\left[1 + (8c_i + 1)^{½}\right]\right\}(D\|\mathbf{a}_i\|)^{-1} - d_i(\|\mathbf{a}_i\|)^{-1}\right]. \tag{13}$$

Expressions (2) and (3) are now substituted into (13):

Proposition 2. The location of maximum item information in the direction of the item discrimination vector $\mathbf{a}_i$ for the M3-PL model is

$$\boldsymbol{\theta}_{\max} = \mathbf{u}_i\left[\ln\left\{0.5\left[1 + (8c_i + 1)^{½}\right]\right\}(D\cdot\mathrm{MDISC}_i)^{-1} + \mathrm{MDIFF}_i\right]. \tag{14}$$

Corollary 2a. The location on the kth dimension where information is maximized is given by

$$\boldsymbol{\theta}_{\max k} = \left[\ln\left\{0.5\left[1 + (8c_i + 1)^{½}\right]\right\}(D\cdot\mathrm{MDISC}_i)^{-1} + \mathrm{MDIFF}_i\right]\cos\alpha_{ki}. \tag{15}$$

Corollary 2b. If it is assumed that there is no guessing, $c_i = 0$, (14) reduces to

$$\boldsymbol{\theta}_{\max} = \mathrm{MDIFF}_i\mathbf{u}_i. \tag{16}$$

The elements of $\boldsymbol{\theta}_{\max}$ in (16), which have the general form of $\mathrm{MDIFF}_i\cos\alpha_{ki}$, are the same as geometric polar coordinates that illustrate graphically item vector plots for the M2-PL model (Reckase, 1985). Thus, item vector plots are geometric representations of $\boldsymbol{\theta}_{\max}$.

Corollary 2c. If it is assumed that there is only one dimension, then the expression in (14) reduces to the well-known formula for $\boldsymbol{\theta}_{\max}$ derived by Birnbaum (1968),

$$\theta_{max} = \ln\left\{0.5\left[1 + (8c_i + 1)^{1/2}\right]\right\}(Da_i)^{-1} + b_i. \tag{17}$$

Conclusion

There are a few potential uses of the results in this study. In multidimensional test design, geometric representations of $\boldsymbol{\theta}_{\max}$ can be used to construct tests according to the M3-PL model. In differential item functioning (DIF) analyses, research has shown that when the angular difference between the direction of best measurement of the item and the test is large (e.g., greater than 30°), items will show DIF (Ackerman & Evans, 1994). Thus, knowing the direction of best measurement for items, which is the direction of the discrimination vector $\mathbf{a}_i$, may be useful in determining what items will exhibit DIF in tests designed according to the M3-PL model. Finally, research-investigating algorithms for multidimensional adaptive tests (e.g., van der Linden, 1999) could perhaps incorporate $\boldsymbol{\theta}_{\max}$ and the direction of measurement in selecting items and then compare this approach to other methods.

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