

# Item Vector Plots for the Multidimensional Three-Parameter Logistic Model

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## Abstract

This is a brief technical note on the creation of item vector plots for the multidimensional 3-parameter logistic model.

## Keywords

multidimensional item response theory, three-parameter logistic model, guessing, item vector plots, theta maximum, graphics

This brief technical note describes how to construct item vector plots for dichotomously scored items fitting the multidimensional three-parameter logistic model (M3PLM). As multidimensional item response theory (MIRT) shows promise of being a very useful framework in the test development life cycle (Ackerman, 1996; Bryant, 2005; Reckase, 1985, 1997), graphical tools that facilitate understanding of the interaction of examinees and items are becoming more important. These informative visual aids include item response surfaces, equiprobability contour plots, and item vector plots. For a description, see Ackerman (1996).

This note is focused on item vector plots used to visually describe a large number of multidimensional items in one graph. Within an item vector plot, each item is typically represented as an arrow projecting from the point of steepest slope in a direction of best measurement; the length of the arrow is represented by the length of the discrimination vector  $\mathbf{a}_i$ . The application of item vector plots has been limited primarily to the multidimensional two-parameter logistic model (M2PLM), and application to more advanced models is necessary to realize MIRT's full potential.

Ackerman (1996) generalized the methods for creating item vector plots from the M2PLM to the M3PLM. While using the formula provided in Reckase (1985) for the M2PLM to calculate the point of steepest slope in the direction of best measurement, Ackerman suggested different representations of arrowheads for the item vectors to reflect various sizes of the pseudoguessing parameter ( $c_i$ ) for the M3PLM. See Figure 4 in the article by Ackerman (1996). Although this was a good initial effort, it is not recommended for creating item vector plots for the M3PLM as guessing is not properly incorporated in the calculations used to create the base of the item

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vector. Thus, the purpose of this technical note is to demonstrate the use of a more appropriate formula by Bryant (2005) for creating item vectors for the M3PLM and illustrate the differences between M2PLM and M3PLM plots.

At the time Ackerman's guidance was made, the mathematical foundation to support its use did not exist. The work by Reckase (1985) provided a framework to support item vector plots for the M2PLM. The foundation for the M3PLM needed to be developed.

Reckase (1997) proposed the following M3PLM:

$$P_i(\boldsymbol{\theta}_j) = c_i + (1 - c_i) \left[ 1 + \exp(-L) \right]^{-1}, \quad (1)$$

where  $L = D(\mathbf{a}_i' \boldsymbol{\theta}_j + d_i)$ , and  $D$  is equal to a scaling constant ( $D = 1$  for the logistic model and  $D = 1.7$  for a reasonable approximation to the normal ogive model);  $\mathbf{a}_i$  is a vector of  $k$  discrimination parameters for item  $i$ ,  $[a_{1i}, a_{2i}, \dots, a_{ki}]'$ , where  $k$  is the number of dimensions;  $\boldsymbol{\theta}_j$  is a vector of  $k$  ability parameters for person  $j$ ,  $[\theta_{1j}, \theta_{2j}, \dots, \theta_{kj}]'$ ;  $d_i$  is a scalar related to difficulty; and  $c_i$  is the pseudoguessing parameter. Bryant (2005) subsequently developed the item information function in a direction defined by a vector of directional cosines associated with the discrimination vector  $\mathbf{a}_i$  for the M3PLM. Bryant also showed that item vector plots for the M2PLM are equal to geometric representations of theta max ( $\boldsymbol{\theta}_{\max}$ ) defined below.

For a specific item  $i$ , it can be easily shown that the direction parallel to  $\mathbf{a}_i$  is also the direction of best measurement for that item. For the M2PLM, the point of steepest slope in the direction of best measurement is also the location in the  $\boldsymbol{\theta}$  space where item information is at a maximum. As in unidimensional IRT, the location on the  $\theta$  scale where information is at a maximum is known as  $\theta_{\max}$  (Hambleton, Swaminathan, & Rogers, 1991). Thus, the point of steepest slope in the direction of best measurement for the M2PLM is equal to  $\boldsymbol{\theta}_{\max}$  in a direction parallel to  $\mathbf{a}_i$ . One major advantage of plotting an item vector using the point of steepest slope in the direction of best measurement for the M2PLM is clear: The item vector indicates not only where the item is most discriminating but also where the item is most informative in the latent ability space, which is at the base of the item vector located at a point on the .5 equiprobability contour (Ackerman, 1996). However, when guessing is present, the point of steepest slope in the direction of best measurement is no longer where the item is most informative. In other words, the point of steepest slope in the direction of best measurement for the M3PLM is not equal to  $\boldsymbol{\theta}_{\max}$  in a direction parallel to  $\mathbf{a}_i$ . Moreover, the base of the item vector is no longer located at a point on the .5 equiprobability contour, which calls current guidance into question.

For the M2PLM, the point of steepest slope in the direction of best measurement and  $\boldsymbol{\theta}_{\max}$  in a direction parallel to  $\mathbf{a}_i$  are both determined when  $P_i(\boldsymbol{\theta}) = .5$ ; this equality is well known for maximizing information for the M2PLM and its unidimensional counterpart. More important is the fact that the base of the item vector is located at the .5 equiprobability contour. For the M3PLM, the point of steepest slope in the direction of best measurement and  $\boldsymbol{\theta}_{\max}$  in a direction parallel to  $\mathbf{a}_i$  are determined under different conditions. The former is determined when  $P_i(\boldsymbol{\theta}) = .5(1 + c_i)$ , and the latter is found when  $P_i(\boldsymbol{\theta}) = .5 \{ \exp[D(\mathbf{a}_i' \boldsymbol{\theta} + d_i)] \}$ ; see Bryant (2005). Of major importance here is the fact that in either case above, the base of the item vector is no longer located at the .5 equiprobability contour as recommended by Ackerman (1996).

For the M3PLM, the base of the item vector representing the point of steepest slope in the direction of best measurement is located at the  $.5(1 + c_i)$  equiprobability contour, and the base of the item vector representing  $\boldsymbol{\theta}_{\max}$  in a direction parallel to  $\mathbf{a}_i$  is located at the  $.5 \{ \exp[D(\mathbf{a}_i' \boldsymbol{\theta}_{\max} + d_i)] \}$  equiprobability contour. These two points are not the same unless  $c_i = 0$ , which yields the  $P_i(\boldsymbol{\theta}) = .5$  equality mentioned earlier for the M2PLM. The point of steepest slope in the direction

of best measurement no longer provides maximum information in the M3PLM context. Thus,  $\theta_{\max}$  in a direction parallel to  $\mathbf{a}_i$  is chosen to represent the item vector in the M3PLM because guessing is properly considered in creating the plot, and information is at a maximum in the direction of best measurement.

If item vector plots for the M3PLM are created using M2PLM formulas provided in Reckase (1985) and the guidance suggested by Ackerman (1996), then coordinates used to create the base of the vectors will be lower than the true values to the degree that guessing is present. Analogous to unidimensional IRT, when guessing is present,  $\theta_{\max}$  for the 3-parameter logistic (3-PL) model is higher (i.e., more difficult) than  $\theta_{\max}$  for the 2-parameter logistic (2-PL) model holding all else constant (Hambleton et al., 1991). This note illustrates the point and provides the proper formula for the creation of item vector plots for the M3PLM. First, the authors review the formulas of  $\theta_{\max}$  for the M3PLM and M2PLM provided by Bryant (2005). Then, they show the error in computing item vector plots for the M3PLM using the M2PLM formula and Ackerman's guidance.

Bryant (2005) proved that the location of maximum information for the M3PLM is

$$\theta_{\max} = \alpha_i \left[ \ln \left\{ .5 \left[ 1 + (8c_i + 1)^{1/2} \right] \right\} (D * MDISC_i)^{-1} + MDIFF_i \right], \quad (2)$$

where  $D = 1$  for the logistic model or 1.7 for a reasonable approximation to the normal ogive model,  $MDISC_i = (\mathbf{a}'_i \mathbf{a}_i)^{1/2}$ ,  $MDIFF_i = -d_i / (\mathbf{a}'_i \mathbf{a}_i)^{1/2}$ , the vector of directional cosines is  $\alpha_i = [\cos \alpha_{1i}, \cos \alpha_{2i}, \dots, \cos \alpha_{ki}]$ , and  $\cos \alpha_{ki} = a_{ki} / MDISC_i$ . For the M3PLM, the location of maximum information on dimension  $k$  of the  $\theta$  space for item  $i$  is

$$\theta_{\max ki} = \cos \alpha_{ki} \left[ \ln \left\{ .5 \left[ 1 + (8c_i + 1)^{1/2} \right] \right\} (D * MDISC_i)^{-1} + MDIFF_i \right]. \quad (3)$$

When there is no guessing, Equation 2 for  $\theta_{\max}$  reduces to the following formula implied by Reckase (1985) for the M2PLM:

$$\theta_{\max} = MDIFF_i \alpha_i. \quad (4)$$

For the M2PLM, the location of maximum information on dimension  $k$  for item  $i$  is

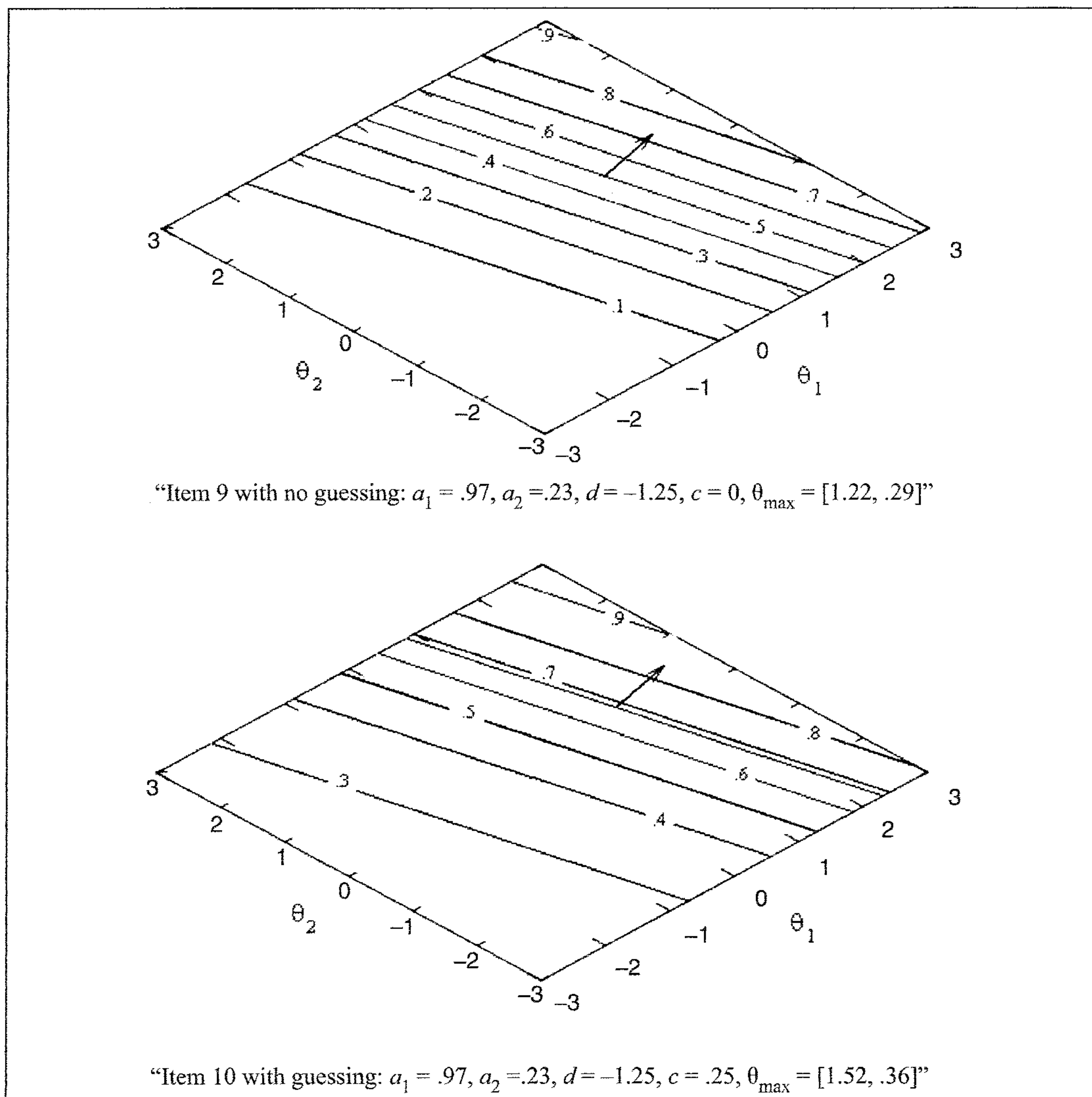
$$\theta_{\max ki} = \cos \alpha_{ki} MDIFF_i = \frac{-d_i a_{ki}}{(\mathbf{a}'_i \mathbf{a}_i)}. \quad (5)$$

Equation 5 in this note is Equation 11 presented in Reckase (1985) for computing item vector plots for the M2PLM. In this report, Equation 4 can be interpreted as "the point of greatest discrimination in the direction  $\alpha_i$  from the origin of the  $\theta$  space" (Reckase, 1985, p. 409). In light of Equation 2, the authors more generally interpret it as the point of greatest *information* in the direction  $\alpha_i$  from the origin of the  $\theta$  space. There are errors when using Equations 4 and 5 to create item vector plots for the M3PLM.

As seen in Equations 2 and 4, there is a substantial difference in the calculations of  $\theta_{\max}$ . In Table 1, the discrepancy in results is clearly illustrated by the Euclidean distance, which is interpreted as the distance between  $\theta_{\max}$  for the M2PLM and M3PLM. The correlation between  $c_i$  and the distance measure is near perfect,  $r = .99, p < .01$ . As guessing increases, the distance between the two  $\theta_{\max}$  calculations increases. The differences in item vectors and equiprobability contour plots are also apparent in Figure 1.

**Table 1.** Location of Maximum Information for Item Pairs With and Without Guessing

Item	$a_{1i}$	$a_{2i}$	$d_i$	$c_i$	M2PLM		M3PLM		Euclidian distance
					$\theta_{\max 1i}$	$\theta_{\max 2i}$	$\theta_{\max 1i}$	$\theta_{\max 2i}$	
1	1.67	1.10	2.50	0.00	-1.04	-0.69	-1.04	-0.69	0.00
2	1.67	1.10	2.50	0.05	-1.04	-0.69	-1.01	-0.66	0.04
3	1.09	1.03	1.88	0.00	-0.91	-0.86	-0.91	-0.86	0.00
4	1.09	1.03	1.88	0.10	-0.91	-0.86	-0.83	-0.79	0.11
5	1.38	0.59	-1.88	0.00	1.15	0.49	1.15	0.49	0.00
6	1.38	0.59	-1.88	0.15	1.15	0.49	1.28	0.55	0.14
7	0.60	0.80	-1.25	0.00	0.75	1.00	0.75	1.00	0.00
8	0.60	0.80	-1.25	0.20	0.75	1.00	0.91	1.21	0.27
9	0.97	0.23	-1.25	0.00	1.22	0.29	1.22	0.29	0.00
10	0.97	0.23	-1.25	0.25	1.22	0.29	1.52	0.36	0.31



**Figure 1.** Item vector and equiprobability contour plots with and without guessing

In the figure, there are two item vector plots with equiprobability contour overlays. The top panel of Figure 1 is Item 9 from Table 1 with no guessing and is plotted according to Ackerman's guidance. The bottom panel of Figure 1 is Item 10 with the exact same parameters as Item 9 with the exception of guessing ( $c_{10} = .25$ ). Along with the information in Table 1, the figures illustrate several notable differences: (a) the items differ graphically in regard to equiprobability contours; (b) although the two items do have the same direction of best measurement, which is 13.34 degrees from  $\theta_1$  and 76.66 degrees from  $\theta_2$ , the vectors differ considerably at the point of maximum information,  $\theta_{\max 9} = [1.22, .29]$  vs.  $\theta_{\max 10} = [1.52, .36]$ ; and (c) the equiprobability contours where the bases of the item vectors are located differ,  $P_9(\theta_{\max}) = .5$  for the M2PLM and  $P_{10}(\theta_{\max}) = .68$  for the M3PLM. If the previous guidance is used to create the item vector plot for Item 10, then the item vector would look exactly like that of Item 9, which is incorrect.

When guessing is present,  $\theta_{\max}$  for the M3PLM (Equation 2) should be used to compute item vectors in the direction of best measurement. The guidance provided by Ackerman (1996) is not recommended because it does not fully consider the impact of the  $c$  parameter in calculating the correct location of the base of the item vector. When guessing is present, the previous guidance will yield item vectors where the base or  $\theta_{\max}$  is lower and less difficult than the correct values of  $\theta_{\max}$  used to compute the item vector base for the M3PLM. This technical note illustrates how to construct item vector plots according to the M3PLM using  $\theta_{\max}$ .

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